Sharp characterization of minimizers (typically) involving interfaces in images

Motivation and Background: Dealing with the enormous, accelerating flood of data is rapidly becoming the single most pressing issue in many scientific and strategic challenges. Much of this data consists of images. Examples include surveillance videos, satellite or airborne imagery, micrographs from a variety of sources, astronomical observations, and tomographic or radiographic measurements. Defensible quantification of uncertainty for the analysis of this data flood requires in turn, not only automation but also precise characterization of the underlying methods. Analysis of scientific data is a case in point. Rigorously defensible inference from scientific images requires optimal extraction of information with quantified uncertainty. Considering this, we see why precisely characterized computational image analysis methods are of utmost importance.

Simply put, for precise inferences from this ever growing stream of data, we need precise characterizations of the underlying numerical methods.

Edges are often among the most important features in an image. If an image is a measurement from a scientific experiment, the interfaces these edges capture can be critically important to understanding what the measurement is telling us. Consequently a large effort has been spent developing methods which process or analyze images in such a way as to preserve, enhance, or extract those edges.

The variational approach to various image analysis tasks minimizes a sum of at least two terms,

$$\min_u \quad F(u) \equiv R(u) + DF(u,f). \tag{1}$$

There is typically a regularization $R(u)$ term encouraging minimizers u to be "nice". By nice we mean that u is smooth or has small high frequency components or belongs to some lower dimensional subspace, etc. And there is typically a data fidelity term $DF(u,f)$ encouraging u to be close to the input image f. (There can of course be other terms which minimize as u approaches various goals defined by the task at hand. We will stick with these two since they are enough for the purposes of this introduction.)

As a concrete example, we consider the functional introduced by Rudin, Osher, and Fatemi in 1992 [6]. They suggested the total variation (TV) seminorm for regularization when denoising images. More precisely, the ROF functional is defined by

$$F(u) = \int |\nabla u| dx + \lambda \int |u-f|^2 dx. \tag{2}$$

The regularization term $\int |\nabla u| dx$ (the TV seminorm) reduces oscillations in u. Very significantly, it is not biased against discontinuities. That is, even though TV prefers less oscillatory images u, it does not prefer smoothed transitions over sharp jumps. We illustrate this now.

Figure 1 shows 4 functions mapping the unit interval to the unit interval. Each function is 0 at 0 and 1 at 1. Each function is monotonically increasing. And the total variation ($\int |\nabla f| dx$) of each is exactly 1. What should be noticed is that TV measures total change (oscillation), not how quickly this change is made. In our example, a function transitioning discontinuously from 0 to 1 has the same total variation as a function transitioning smoothly. The second (data fidelity) term on the right hand side of Equation 2 measures how close u,

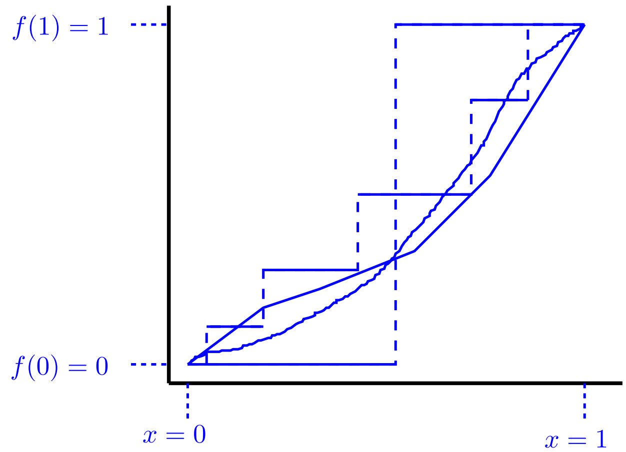

Figure 1: Each of the four functions has the same TV seminorm.

the candidate denoised image, is to the input data f.

From a probabilistic viewpoint, (ignoring complications due to infinite dimensionality), we might try maximizing the probability of u given the data f as an approach to denoising. This probability, $p(u|f)$ called the posterior, can be maximized using Bayes relation,

$$p(u|f) \sim p(f|u)p(u),$$

where $p(f|u)$ is the likelihood (probability of data given true image is u) and $p(u)$ is the prior distribution on true images. If we take

$$\begin{aligned} p(u) &\sim e^{-\int |\nabla u| dx} \text{ (a reasonable smoothing energy), and} \\ p(f|u) &\sim e^{-\lambda \int |u-f|^2 dx} \text{ (Gaussian measurement errors)} \end{aligned}$$

then, upon taking the negative logarithms and ignoring constant terms, we get

$$\begin{aligned} -log(p(u)) &= \int |\nabla u| dx \\ -log(p(f|u)) &= \lambda \int |u - f|^2 dx. \end{aligned}$$

We now want to *minimize* $F(u)$, given by

$$F(u) \equiv -log(p(u|f)) = \int |\nabla u| dx + \lambda \int |u - f|^2 dx.$$

But this is exactly the ROF functional.

This explains identifications sometimes made:

$$\begin{aligned} \text{regularization term} &= \text{the prior, and} \\ \text{data fidelity term} &= \text{the likelihood function.} \end{aligned}$$

The main point to take away from this probabilistic detour is that: 1) the regularization term quantifies, independently of measurement, how reasonable a proposed denoised image u is, and 2) the data fidelity term quantifies how reasonable u is as a measurement of f.

With this very brief look at variational functionals, we move on to the critical questions we intend to address.

What are the pressing questions for TV regularized variational methods?

1) How close is a computation to convergence? Because we typically cannot compute minimizers analytically, various numerical methods are used to approximate minimizers. How do we know when a particular computation has converged? How does convergence depend on input data? For instance, can we relate local measures of regularity to local convergence rates? For scientific images, precise inferences can depend critically on convergence.

2) What are the precise geometric properties of the minimizers? The TV seminorm tries to minimize the lengths of level sets of u while the data fidelity term attempts to enforce close agreement with the input image. These two driving forces, typically in opposition, balance each other at the minimizer. What does this balance tell us about the geometry of the minimizer? Can we use this information to build exact minimizers? (Previous work on exact minimizers is limited to 1-dimensional cases (ROF), discrete cases (L^1TV and ROF) or very special cases in 2-D (ROF). See references in [3] for details.)

3) Can we create a hybrid analytic/numerical approach to minimization by exploiting 2)? It would seem that exact solutions for a rich enough family of input images f should enable us to construct a hybrid scheme giving us better control over solution errors. What are the obstacles, if any, to doing this?

New Science and Research Plan:

Using tools from geometric measure theory, we have very recently [5,1,2] developed insights into the precise nature of minimizers for the L^1TV functional, defined by

$$F(u) = \int |\nabla u| dx + \lambda \int |u - f| dx, \tag{3}$$

(note the L^1 data fidelity term) and for the ROF functional, which you recall is given by

$$F(u) = \int |\nabla u| dx + \lambda \int |u - f|^2 dx.$$

(See [3] for more background on the L^1TV functional.) As a result, we can now construct whole families of non-trivial, exact minimizers for the L^1TV and ROF functionals.

Geometric measure theory, though very natural and intuitively appealing, can be quite intricate and technically demanding. Fortunately, although we used significant parts of geometric measure theory, the insights gained are simple to state and understand. We briefly skim these insights.

In what follows, we will work in $\mathbb{R}^2$ which is sufficient for most image analysis applications. Ω and Σ will be subsets of $\mathbb{R}^2$, χ_E will denote the characteristic function of $E \subset \mathbb{R}^2$ (i.e. $\chi_E = 1$ on E and 0 otherwise), B_r will be a ball of radius r, and ∂E will denote the boundary of E (glossing over many issues which lead to various refinements like the reduced boundary and the measure theoretic boundary). We use the fact that $f = \chi_\Omega$ implies there is an L^1TV minimizer $u = \chi_\Sigma$, to refer to input data by either $f = \chi_\Omega$ or Ω and to the minimizer by either $u = \chi_\Sigma$ or Σ. Ω will always correspond to input data and Σ will always correspond to the minimizer.

Insight #1: *For the L^1TV functional, $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda}} \subset \Sigma$. Likewise, $B_{\frac{2}{\lambda}} \subset \Omega^c \rightarrow B_{\frac{2}{\lambda}} \subset \Sigma^c$.* This permits a first pass at determining the exact solution: $\partial\Sigma$ is in the envelope between the $B_{\frac{2}{\lambda}}$ outside of Ω and the $B_{\frac{2}{\lambda}}$ inside Ω. This is illustrated in the Figure 2. (Related tidbit: $\Omega \subset B_{\frac{2}{\lambda}-\epsilon}$, $\epsilon > 0 \rightarrow \Sigma = \{$the empty set$\}$ is the unique minimizer.)

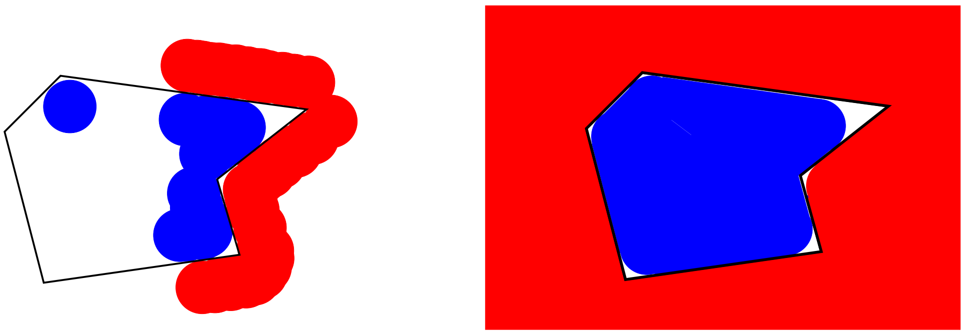

Figure 2: As a result of insight #1, we know that the boundary of the minimizer Σ lies between the red(exterior) and blue(interior) balls.

Insight #2: *For the L^1TV functional, $B_{\frac{2}{\lambda}}$ almost in $\Omega \rightarrow B_{\frac{2}{\lambda}}$ almost in Σ.* Think of noise removing small parts of Ω or adding small islands of noise outside of Ω. This insight says the minimizer is almost not affected by that noise. Therefore, Figure 2 is close to the right picture for insight #2.

Insight #3: For the L^1TV and the ROF functionals, *the first variation computed for carefully chosen curves in the space of images yields precise formulas prescribing the curvature of minimizer level sets.* L^1TV example: when the the boundary of the minimizer Σ diverges from the boundary of the input set Ω, its curvature is exactly λ (i.e. it is an arc of a circle of radius $1/\lambda$). Additionally, the boundary of Σ in the interior of Ω bulges out, while the boundary of Σ exterior to Ω bulges in. Together with #1 above, this permits us to generate exact solutions as shown in center subfigure of Figure 3.

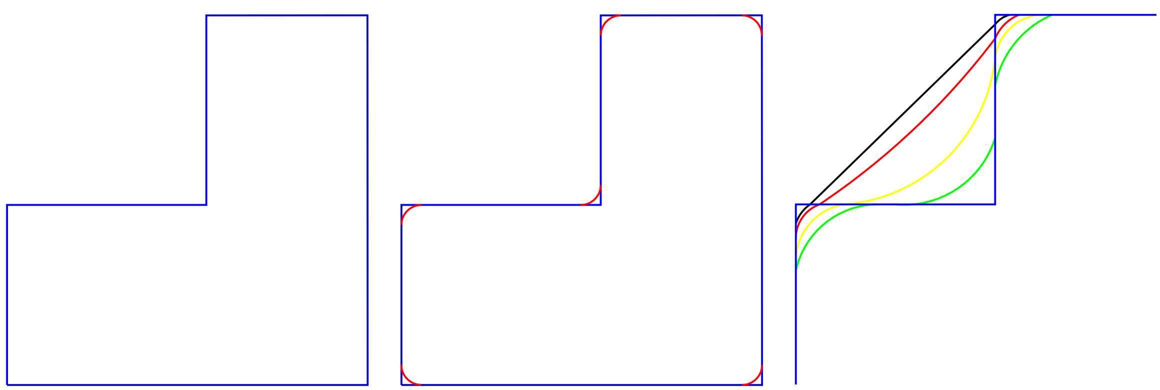

Figure 3: Left to right: $\partial\Omega$, $\partial\Sigma$ (L^1TV) with $\partial\Omega$ superimposed for comparison, and parts of a few level sets of the ROF minimizer, with part of $\partial\Omega$ superimposed for comparison.

What can we do with these insights? A great deal! Using families of exact solutions, we can, for the first time, understand the precise convergence properties of these important methods in image analysis. Our ability to generate these families of exact solutions also suggests development of a hybrid analytic/numerical approach to computing minimizers given input data f. Using the same tools from geometric measure theory , we can develop similar insights into the precise nature of minimizers for related, widely used functionals. Finally, we can leverage detailed knowledge obtained into development of new variational methods tuned specifically for the data at hand. **It is precisely these things that we now propose to do.**

(I) We Propose: to use carefully selected, nontrivial exact minimizers to precisely characterize convergence of numerical methods in use. These include (1) gradient descent, (2) lagged diffusivity fixed point iterations, and (3) the Chambolle duality method. We will then be able to quantify the uncertainty introduced or removed by the use of these algorithms.

(II) We Propose: to develop a hybrid analytic/numerical scheme for TV based methods. We will begin (IIa) with the L^1TV functional. After success there, we will (IIb) tackle the more complicated ROF functional. We expect a speedup, sometimes significant, in comparison to other methods in use. Additionally, precise control of solution errors should be easier to implement.

(III) We Propose: to generate similar sharp characterizations for a a generalized functional given by,

$$F_\phi(u) = \int \phi(\nabla u)dx + \lambda \int |Pu - f|^p dx, \tag{4}$$

where ϕ is a positive 1-homogeneous function, P is a linear operator, and p = 1 or 2. The choice of ϕ is used to build directional preferences into $F_\phi(u)$. An example is crystal growth where we choose ϕ based on the Wulff shape. See [4] for more details on such a model.

Milestones: Convergence characterization (subproject I) will occupy the first year and approximately the first quarter of the second year. Development of a hybrid method for the L^1TV functional (Subproject IIa) will be completed sometime after the first year, but before the halfway point of the project. Development of a hybrid method for the ROF functional (Subproject IIb) will begin after the completion of IIa and be completed by the end of the third year. Theory for the generalized functional (subproject III) will begin immediately and will extend throughout the entire three year period.

Impact for Science and LANL Programs:

It is universally recognized that quantification of uncertainty in analysis of experiments and simulations has too long been ignored or inadequately dealt with. Understanding the precise nature of algorithmic convergence and the exact form of minimizers permits us to know 1) how close we are to a minimizer and 2) what our methods of analysis are inserting, removing or enhancing in the data. These are key steps in quantifying the propagation of uncertainty along the analysis path.

Example: Wave Collider Experiments. In this experiment, two cylinders of high explosive (HE) are joined together on end. Opposite ends are ignited. The burn fronts meet and pass through each other. Continuing on, they form a lens shaped region defined by the oppositely moving fronts. This can be seen clearly in the left-most subfigure of Figure 4. The total variation inversion (right) is a great improvement over the typical inversion (center). But for the purposes of precise quantification of errors and propagation of uncertainties, we must have a precise understanding of what the analysis methods do to critical features such as the edges.

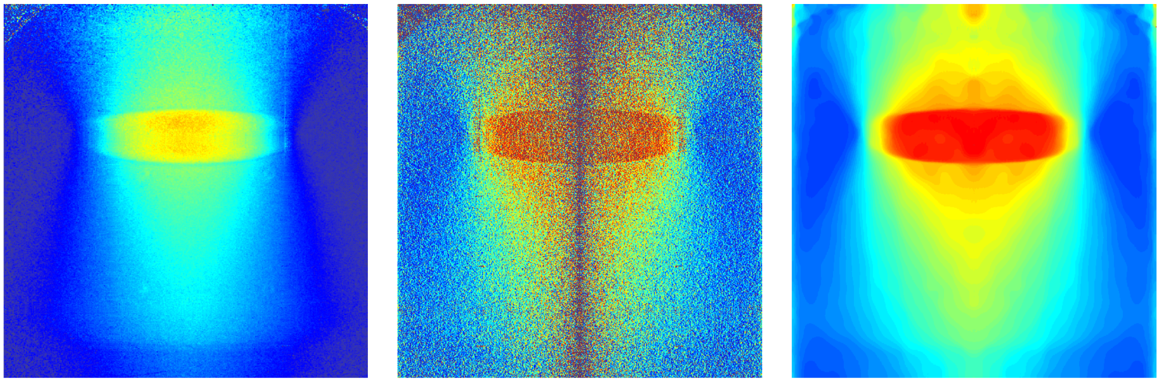

Figure 4: Left to right: raw radiograph of HE wave collider experiment, unregularized Abel inversion, and TV regularized Abel inversion.

Given recent, increased emphasis on quantification of uncertainty, our proposal to provide significant new tools for that purpose is right on target. Precise, 2-dimensional characterization for general minimizers is completely new for variational methods in image analysis, promising to yield significant advances both theoretically and computationally.

Summary: To quantify uncertainties in an analysis path, one must have precise characterizations of the methods used. Our proposed study and exploitation of sharp characterizations of minimizers give us several tools that move us a great deal closer to the goal of rigorously defensible quantification of uncertainty. In addition, the understanding gained can be leveraged into the construction of new, more powerful methods of computation and analysis.

(1) W. K. Allard. On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization; I. Theory. Preprint, 2006. http://www.math.duke.edu/~wka/papers/bv.pdf

(2) W. K. Allard. On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization; II. Examples. Preprint, 2006. http://www.math.duke.edu/~wka/papers/examples.pdf

(3) T. F. Chan and S. Esedoḡlu. Aspects of total variation regularized L^1 function approximation. SIAM J. Appl. Math., 65(5):1817-1837, 2005.

(4) S. Esedoḡlu and S. J. Osher. Decomposition of images by the anisotropic Rudin-Osher-Fatemi model. Commun. Pure Appl. Math., 57:1609-1626, 2004.

(5) S. Esedoḡlu and K. R. Vixie. Some properties of minimizers for the L^1TV functional. In preparation. http://ddma.lanl.gov/~vixie/private/L1TV-rough-draft.pdf

(6) L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. Physica D, 60(1-4):259-268, November 1992.

Curriculum Vitae: Kevin R. Vixie

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Relevant Publications :

1. *Nonlinear regularizations of TV based PDEs for image processing* (with A Bertozzi, J Greer, and SJ Osher), in Contemporary Mathematics vol. 371, AMS 2005
2. *Some properties of minimizers for the* $L^1 TV$ *Functional* (with S Esedoḡlu), in preparation, 2006
3. *Abel inversion using total-variation regularization* (with TJ Asaki, R Chartrand,and BE Wohlberg), Inverse Problems, no. 21, pp. 1895-1903, 2005 http://math.lanl.gov/Research/Publications/Docs/asaki-2005-abel.pdf
4. *Abel inversion using total variation regularization: Applications* (with TJ Asaki, PR Campbell, R Chartrand, CE Powell, and BE Wohlberg) Submitted, 2005
5. *Defensible metrics and merit functions* (with TJ Asaki) in "Los Alamos Science", 2005 LA-UR-04-8498.

Selected Organizational Activities :

* **2002 LANL Radiography Analysis and Simulation Tools Workshop:** On organizing committee.
* **2002 LANL Image Analysis Workshop:** Main organizer, December 2-6 2002 http://ddma.lanl.gov/public/conference/2002-lwia/
* **2003-2005 IPAM Research in Industrial Projects for Students (RIPS):** Industry sponsor, http://www.ipam.ucla.edu/programs/rips2005/
* **2005 IPAM Graduate Summer School, Chair and lecturer:** "Intelligent Extraction of Information from Graphs and High Dimensional data." http://www.ipam.ucla.edu/programs/gss2005/

Current Projects: Data Driven Modeling and Analysis (DDMA) team projects include (1) several on metrics and regularization for validation of weapons codes, (2) radiographic inversions for both Proton and X-ray measurements, (3) development of novel metrics for high-dimensional data (LDRD), (4) an intelligence community project (a difficult inverse problem), (5) development of IDA, a flexible, extensible suite of algorithms for the analysis of image data, (6) part of the muon radiography LDRD-DR, and (7) many papers looking at various theoretical and applied aspects of all of this work. My personal research focus is on geometric measure theory, high-dimensional geometry, and algorithms based on these tools for challenging data problems. I raise or help raise a large portion of the DDMA budget of 2̃.5M$/year.

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Doctoral Dissertation:
On Boundary Regularity for Plateau's Problem,
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Employment:
Research Assistant, Brown University, 1967-1968
Instructor, Princeton University, 1968-1969
Lecturer, 1969-1970
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Fellowships, Honors, etc.:
NSF Cooperative Graduate Fellowship, 1968-1970
Alfred P. Sloan Foundation Fellowship, 1970-1972
Invited Speaker at 1973 Annual Meeting of the
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Invited Speaker at 1974 International Congress of Mathematicians
Managing Editor, Duke Mathematical Journal 1983-1985
Co-chairman, 1984 American Mathematical Society Summer Institute
Chairman, Mathematics Department, Duke University, 1985-1986

Selected Publications:

1. *On the first variation of a varifold*, Ann. of Math. **95** (1972), 417-491.
2. *On the first variation of a varifold: Boundary behavior*, Ann. of Math. **101** (1975), 418-446.
3. *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, (with F. J. Almgren, Jr.), Ann. of Math. **113** (1981), 215-265.
4. (with J.A. Trangenstein) *On the performance of a distributed object oriented adaptive mesh refinement code*, Preprint, October 1997, http://www.math.duke.edu/~wka/
5. *On the regularity and Curvature properties of level sets of minimizers for denoising models using total variation regularization; I. Theory*, Preprint, http://www.math.duke.edu/~wka/papers/bv.pdf
6. *On the regularity and curvature properties of level sets of minimizers for denoising models using total variation regularization; II. Examples*, Preprint, http://www.math.duke.edu/~wka/papers/examples.pdf
7. *Computing length and areas of boundaries of regions given a binary representation*, In preparation.

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Employment

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Professional Preparation

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1998 M.S. in Mathematics, Courant Institute
1996 Sc.B. in Mathematics, Brown University. Magna cum laude, honors in mathematics.

Appointments

2005-Present Assistant Professor. Mathematics, University of Michigan Ann Arbor
2002-2005 CAM Assistant Professor. Mathematics, UCLA
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Research Interests

Image processing, computer vision; partial differential equations, calculus of variations; convergence of numerical approximations.

Honors

04/2001 Kurt O. Friedrichs Prize for an Outstanding Dissertation, *Courant Institute*
05/1996 David Howell Prize for Excellence in Mathematics, *Brown University*
1992-1996 Granoff International National Scholarship, *Brown University*

Selected Publications

1. S. Esedoglu. An analysis of the Perona-Malik scheme. *Comm. Pure Appl. Math.* **54** (2001), pp. 1442 – 1487.
2. S. Esedoglu. Stability properties of the Perona-Malik scheme. *To appear in SIAM J. Numer. Anal.*
3. S. Esedoglu, S. J. Osher. Decomposition of images by the anisotropic Rudin - Osher - Fatemi model. *Comm. Pure Appl. Math.* **57** (2004), pp. 1609 – 1626.
4. T. F. Chan, S. Esedoglu. Aspects of total variation regularized L^1 function approximation. *SIAM J. Appl. Math.* **65**:5 (2005), pp. 1817 – 1837.
5. T. F. Chan, S. Esedoglu, M. Nikolova. Algorithms for finding global minimizers of denoising and segmentation models *To appear in SIAM J. Appl. Math.*
6. S. Esedoglu, Y.-H. Tsai. Threshold dynamics for the piecewise constant Mumford – Shah functional. *J. Comput. Phys.* **211**:1 (2006), pp. 367 – 384.